

# On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions: the critical case

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## Abstract

In [3], the authors proved that uniqueness holds among solutions whose exponentials are  $L^p$  with  $p$  bigger than a constant  $\gamma$  ( $p > \gamma$ ). In this paper, we consider the critical case:  $p = \gamma$ . We prove that the uniqueness holds among solutions whose exponentials are  $L^\gamma$  under the additional assumption that the generator is strongly convex.

**Key words and phrases.** Backward stochastic differential equations, generator of quadratic growth, unbounded terminal condition, uniqueness result.

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## 1 Introduction

Since the seminar paper [5], backward stochastic differential equations (BSDEs in short for the remaining of the paper) have found many applications in various domains. A lot of efforts have been made in order to study the well posedness of these equations. Quadratic BSDEs is a kind of BSDE which has attracted particular attention recently and it is the subject of the paper.

In this article, we consider the following quadratic BSDE

$$Y_t = \xi - \int_t^T g(Z_s) ds + \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (1.1)$$

where the generator  $g$  is a continuous real function that is convex and has a quadratic growth with respect to the variable  $z$ . Moreover  $\xi$  is an unbounded random variable (see e.g. [4] for the case of quadratic BSDEs with bounded terminal conditions). Let us recall that, in the previous equation, we are looking for a pair of processes  $(Y, Z)$  which is required to be adapted with respect to the filtration generated by the  $\mathbb{R}^d$ -valued Brownian motion  $W$ .

In order to state the main result of this paper, let us suppose that there exists a constant  $\gamma > 0$  such that

$$\xi^+ \in L^1, \exp(-\gamma\xi) \in L^1 \text{ and } 0 \leq g(z) \leq \frac{\gamma}{2} |z|^2.$$

By a localization procedure similar to that in [1], we prove easily that the BSDE (1.1) has at least a solution  $(Y, Z)$  such that  $e^{-\gamma Y}$  and  $Y$  belong to the class (D).

Concerning the uniqueness issue, in [2], the authors proved that the uniqueness holds among solutions whose exponentials are in any  $L^p$ . In [3], the authors proved that the uniqueness holds among solutions whose exponentials are in  $L^p$  for a given  $p > \gamma$ , i.e.

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{pY_t^-} \right] < \infty.$$

However, if we take  $g(z) = \frac{\gamma}{2} |z|^2$ , then it is easy to see that for the associated BSDE, the uniqueness holds among solutions  $(Y, Z)$  such that  $e^{-\gamma Y}$  belongs to the class (D). It suffices to note that if  $(Y, Z)$  is a solution such that  $e^{-\gamma Y}$  belongs to the class (D), then  $e^{-\gamma Y}$  is a uniformly integrable martingale and

$$Y_t = -\frac{1}{\gamma} \ln \mathbb{E} [e^{-\gamma \xi} | \mathcal{F}_t].$$

So the aim of this paper is to study the uniqueness of solution of BSDE (1.1) in the critical case:  $p = \gamma$ . We prove that the BSDE (1.1) has a unique solution  $(Y, Z)$  such that  $e^{-\gamma Y}$  belongs to the class (D) under the additional assumption that the generator  $g$  is strongly convex. We do not know if this result stays true without this additional assumption.

The paper is organized as follows. Next section is devoted to an existence result, section 3 contains a useful property for solutions and the last section is devoted to our main uniqueness result.

Let us close this introduction by giving notations that we will use in all the article. For the remaining of the paper, let us fix a nonnegative real number  $T > 0$ . First of all,  $(W_t)_{t \in [0, T]}$  is a standard Brownian motion with values in  $\mathbb{R}^d$  defined on some complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ .  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration of the Brownian motion  $W$  augmented by the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ .

As mentioned before, we will deal only with real valued BSDEs which are equations of type (1.1). The function  $g$  is called the generator and  $\xi$  the terminal condition. Let us recall that a generator is a function  $\mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$  which is measurable with respect to  $\mathcal{B}(\mathbb{R}^{1 \times d})$  and a terminal condition is simply a real  $\mathcal{F}_T$ -measurable random variable. By a solution to the BSDE (1.1) we mean a pair  $(Y_t, Z_t)_{t \in [0, T]}$  of predictable processes with values in  $\mathbb{R} \times \mathbb{R}^{1 \times d}$  such that  $\mathbb{P}$ -a.s.,  $t \mapsto Y_t$  is continuous,  $t \mapsto Z_t$  belongs to  $L^2(0, T)$ ,  $t \mapsto g(Z_t)$  belongs to  $L^1(0, T)$  and  $\mathbb{P}$ -a.s.  $(Y, Z)$  verifies (1.1).

For any real  $p \geq 1$ ,  $S^p$  denotes the set of real-valued, adapted and càdlàg processes  $(Y_t)_{t \in [0, T]}$  such that

$$\|Y\|_{S^p} := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right]^{1/p} < +\infty.$$

$\mathcal{M}^p$  denotes the set of (equivalent class of) predictable processes  $(Z_t)_{t \in [0, T]}$  with values in  $\mathbb{R}^{1 \times d}$  such that

$$\|Z\|_{\mathcal{M}^p} := \mathbb{E} \left[ \left( \int_0^T |Z_s|^2 ds \right)^{p/2} \right]^{1/p} < +\infty.$$

We also recall that  $Y$  belongs to the class (D) as soon as the family  $\{Y_\tau : \tau \leq T \text{ stopping time}\}$  is uniformly integrable.

For any convex function  $f : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ , we denote  $f^*$  the Legendre-Fenchel transform of  $f$  given by

$$f^*(q) = \sup_{z \in \mathbb{R}^{1 \times d}} (zq - f(z)), \quad \forall q \in \mathbb{R}^d.$$

We also denote  $\partial f$  the subdifferential of  $f$ . We recall that the subdifferential of  $f$  at  $z_0$  is the non-empty convex compact set of elements  $u \in \mathbb{R}^d$  such that

$$f(z) - f(z_0) \geq (z - z_0)u, \quad \forall z \in \mathbb{R}^{1 \times d}.$$

Finally, for any predictable process  $(q_t)_{t \in [0, T]}$  such that  $\int_0^T |q_s|^2 ds < +\infty$   $\mathbb{P}$ -a.s., we denote  $\mathcal{E}(q)$  the Doléans-Dade exponential

$$\left( \exp \left( \int_0^t q_s dW_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right) \right)_{t \in [0, T]}.$$

## 2 An existence result

Let us begin by giving some assumptions used in this paper.

**Assumption A.** There exists a constant  $\gamma > 0$  such that

1.  $\xi^+ \in L^1$  and  $\exp(-\gamma\xi) \in L^1$ ,
2.  $g : \mathbb{R}^d \rightarrow \mathbb{R}^+$  is a convex function that satisfies
  - (a)  $g(0) = 0$ ,
  - (b) there exists a constant  $C_1 \geq 0$  such that  $\forall z \in \mathbb{R}^{1 \times d}$ ,

$$g(z) \leq C_1 + \frac{\gamma}{2} |z|^2.$$

**Assumption B.** There exist two constants  $\varepsilon > 0$  and  $C_2 \geq 0$  such that  $\forall z, z' \in \mathbb{R}^{1 \times d}, \forall s \in \partial g(z')$ ,

$$g(z) - g(z') - (z - z')s \geq \frac{\varepsilon}{2} |z - z'|^2 - C_2.$$

### Remark 2.1

- If  $g$  is a  $C^2$  function then assumption B is equivalent to the assumption: there exist  $R \geq 0$  and  $\varepsilon > 0$  such that for all  $z \in \mathbb{R}^{1 \times d}$  with  $|z| > R$ , we have  $g''(z) \geq \varepsilon Id$ .
- For a general convex generator  $g$  with quadratic growth it is easy to modify the terminal condition and the probability to obtain a new generator  $\tilde{g} : \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}^+$  such that assumption A.2. holds true.

The aim of this section is to show the existence of solutions under the assumption A, using a localization method.

**Theorem 2.2** *Let us assume that assumption A holds. Then the BSDE (1.1) has at least a solution  $(Y, Z)$  such that:*

$$-\frac{1}{\gamma} \ln \mathbb{E} \left[ e^{\gamma C_1 T} e^{\gamma \xi^-} \middle| \mathcal{F}_t \right] \leq Y_t \leq \mathbb{E} [\xi | \mathcal{F}_t].$$

In particular,  $e^{-\gamma Y}$  and  $Y$  belong to the class (D).

**Proof of Theorem 2.2.** To show this existence result we use the same classical localization argument as Briand and Hu in [1]. Let us fix  $n, p \in \mathbb{N}^*$  and set  $\xi^{n,p} = \xi^+ \wedge n - \xi^- \wedge p$ . Then it is known from [4] that the BSDE

$$Y_t^{n,p} = \xi^{n,p} - \int_t^T g(Z_s^{n,p}) ds + \int_t^T Z_s^{n,p} dW_s, \quad 0 \leq t \leq T,$$

has a unique solution  $(Y^{n,p}, Z^{n,p}) \in \mathcal{S}^\infty \times \mathcal{M}^2$ . By applying Theorem 2 in [1], we have the estimate

$$-\frac{1}{\gamma} \ln \mathbb{E} [\phi_t(-\xi^{n,p}) | \mathcal{F}_t] \leq Y_t^{n,p}$$

where  $(\phi_t(z))_{t \in [0, T]}$  stands for the solution to the integral equation

$$\phi_t(z) = e^{\gamma z} + \int_t^T H(\phi_s(z)) ds, \quad 0 \leq t \leq T,$$

with

$$H(p) = C_1 \gamma p \mathbb{1}_{[1, +\infty[}(p) + C_1 \gamma \mathbb{1}_{]-\infty, 1]}(p).$$

It is noticed in [1] that  $\phi_t(z) = e^{\gamma C_1(T-t)} e^{\gamma z}$  when  $z \geq 0$  and  $z \mapsto \phi_t(z)$  is an increasing continuous function. Thus, we have

$$-\frac{1}{\gamma} \ln \mathbb{E} \left[ e^{\gamma C_1 T} e^{\gamma \xi^-} \middle| \mathcal{F}_t \right] \leq -\frac{1}{\gamma} \ln \mathbb{E} \left[ e^{\gamma C_1 T} e^{\gamma (\xi^{n,p})^-} \middle| \mathcal{F}_t \right] \leq -\frac{1}{\gamma} \ln \mathbb{E} [\phi_t(-\xi^{n,p}) | \mathcal{F}_t] \leq Y_t^{n,p}.$$

Moreover,  $g$  is a nonnegative function, so

$$Y_t^{n,p} = \mathbb{E} \left[ \xi^{n,p} - \int_t^T g(Z_s^{n,p}) ds \middle| \mathcal{F}_t \right] \leq \mathbb{E} [\xi^{n,p} | \mathcal{F}_t] \leq \mathbb{E} [\xi^+ | \mathcal{F}_t].$$

We remark that

$$\forall t \in [0, T], \quad Y_t^{n,p+1} \leq Y_t^{n,p} \leq Y_t^{n+1,p},$$

and we define  $Y^p = \sup_{n \geq 1} Y_t^{n,p}$  so that  $Y_t^{p+1} \leq Y_t^p$  and  $Y_t = \inf_{p \geq 1} Y_t^p$ . By the dominated convergence theorem, we have

$$-\frac{1}{\gamma} \ln \mathbb{E} \left[ e^{\gamma C_1 T} e^{\gamma \xi^-} \middle| \mathcal{F}_t \right] \leq -\frac{1}{\gamma} \ln \mathbb{E} [\phi_t(-\xi) | \mathcal{F}_t] \leq Y_t \leq \mathbb{E} [\xi | \mathcal{F}_t],$$

and in particular, we remark that  $\lim_{t \rightarrow +\infty} Y_t = \xi = Y_T$ . Arguing as in [1] with a localization argument, we can show that there exists a process  $Z$  such that  $(Y, Z)$  solves the BSDE (1.1). Finally, since processes  $t \mapsto \mathbb{E} \left[ e^{\gamma C_1 T} e^{\gamma \xi^-} \middle| \mathcal{F}_t \right]$  and  $t \mapsto \mathbb{E} [\xi | \mathcal{F}_t]$  belong to the class (D), we conclude that  $e^{-\gamma Y}$ ,  $Y^+$  and so  $Y$  belong to the class (D).  $\square$

### 3 A uniform integrability property for solutions

In this part we will show the following proposition.

**Proposition 3.1** *We assume that assumption A holds true. Let us consider  $(Y, Z)$  a solution of the BSDE (1.1) such that  $Y$  and  $e^{-\gamma Y}$  belong to the class (D). Then, for all predictable process  $(q_s)_{s \in [0, T]}$  with values in  $\mathbb{R}^d$  and such that  $q_s \in \partial g(Z_s)$  for all  $s \in [0, T]$ ,  $\mathcal{E}(q)$  is a uniformly integrable process and defines a probability  $\mathbb{Q} \sim \mathbb{P}$ .*

**Proof of Proposition 3.1.** Let us start the proof by giving a simple lemma.

**Lemma 3.2** *The family of random variables  $\{e^{\gamma X} | X \in \mathcal{H}\}$  is uniformly integrable if and only if there exists a function  $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $k(x) \rightarrow +\infty$  when  $x \rightarrow +\infty$ , and*

$$\sup_{X \in \mathcal{H}} \mathbb{E}[K(X^+)] < +\infty,$$

with  $K(x) = \int_0^x k(t) e^{\gamma t} dt$ . Moreover, we can assume without restriction that  $k \in C^\infty$ ,  $k(0) = \gamma$  and  $k'(x) > 0$  for all  $x \in \mathbb{R}^+$ .

**Proof of Lemma 3.2.** We only prove the nontrivial implication. Firstly, let us remark that  $\{e^{\gamma X} | X \in \mathcal{H}\}$  is uniformly integrable if and only if  $\{e^{\gamma X^+} | X \in \mathcal{H}\}$  is also uniformly integrable, so we can assume that  $\mathcal{H}$  is a family of positive random variables. Now we apply the de la Vallée-Poussin theorem: there exists a nondecreasing function  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is a constant function on each interval  $[n, n+1[$  for  $n \in \mathbb{N}$ , that satisfies  $g(x) \rightarrow +\infty$  when  $x \rightarrow +\infty$  and such that

$$\sup_{X \in \mathcal{H}} \mathbb{E}[G(e^{\gamma X})] < +\infty,$$

with  $G(x) = \int_1^x g(t) dt$ . Then, it is simple to consider a smooth approximation  $\tilde{g}$  of  $g$  such that  $\tilde{g}(1) = 1$ ,  $\tilde{g}'(x) > 0$  for all  $x \in [1, +\infty[$  and  $g + 1 - g(1) \leq \tilde{g} \leq g + C$ . This function  $\tilde{g}$  also satisfies  $\tilde{g}(x) \rightarrow +\infty$  when  $x \rightarrow +\infty$  and

$$\sup_{X \in \mathcal{H}} \mathbb{E}[\tilde{G}(e^{\gamma X})] < +\infty,$$

with  $\tilde{G}(x) = \int_1^x \tilde{g}(t) dt$ . A simple calculus gives us

$$\tilde{G}(e^{\gamma x}) = \int_0^t \tilde{g}(e^{\gamma u}) \gamma e^{\gamma u} du$$

and so we just have to set  $k(x) = \gamma \tilde{g}(e^{\gamma x})$  to conclude the proof.  $\square$

Now, let us apply the previous lemma in our situation: since we consider a solution  $(Y, Z)$  such that  $e^{-\gamma Y}$  belongs to the class (D), then there exists a function  $k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  given by Lemma 3.2 such that

$$\sup_{0 \leq \tau \leq T, \text{ stopping time}} \mathbb{E}[K(Y_\tau^-)] < +\infty, \quad (3.1)$$

with  $K(x) = \int_0^x k(t)e^{\gamma t} dt$ . We define

$$\Psi_0(x) = e^{\gamma x} - \gamma x - 1 = \int_0^x \gamma(e^{\gamma u} - 1) du \quad \text{and} \quad \Psi(x) = \int_0^x k(u)(e^{\gamma u} - 1) du.$$

Since  $\Psi_0$  and  $\Psi$  are convex functions we can also consider their dual functions.  $\Phi_0(x) = \left(\frac{x}{\gamma} + 1\right) \ln\left(\frac{x}{\gamma} + 1\right) - \frac{x}{\gamma}$  is the dual function of  $\Psi_0$  since  $\Phi'_0(x) = \frac{1}{\gamma} \ln\left(\frac{x}{\gamma} + 1\right)$  is the inverse function of  $\Psi'_0$ . Moreover, the dual function of  $\Psi$  is given by  $\Phi(x) = \int_0^x \Phi'(u) du$  with  $\Phi'$  the inverse function of  $\Psi'$ .

Now we consider a predictable process  $(q_s)_{s \in [0, T]}$  with values in  $\mathbb{R}^d$  and such that  $q_s \in \partial g(Z_s)$  for all  $s \in [0, T]$ . Firstly let us show that  $s \mapsto q_s$  belongs to  $L^2(0, T)$   $\mathbb{P}$ -a.s.. Since assumption A.2 holds true for  $g$ , then  $g^*$  satisfies

$$g^*(q) \geq -C + \frac{1}{2\gamma} |q|^2 \quad \text{and} \quad g^*(0) = 0, \quad (3.2)$$

and thus,

$$\int_0^T |q_s|^2 ds \leq C + C \int_0^T g^*(q_s) ds = C + C \int_0^T (Z_s q_s - g(Z_s)) ds \leq C + C \int_0^T |Z_s q_s| ds + C \int_0^T |Z_s|^2 ds.$$

Moreover, since  $q_s \in \partial g(Z_s)$  we have

$$Z_s q_s = (2Z_s - Z_s) q_s \leq g(2Z_s) - g(Z_s)$$

and

$$-Z_s q_s = (0 - Z_s) q_s \leq g(0) - g(Z_s).$$

So we finally obtain

$$\int_0^T |q_s|^2 ds \leq C + C \int_0^T |Z_s|^2 ds < +\infty \quad \mathbb{P}\text{-a.s..}$$

Now let us show that  $\mathcal{E}(q)$  is a uniformly integrable martingale. We start by defining the stopping time

$$\tau_n = \inf \left\{ t \in [0, T] : \sup \left( \int_0^t |q_s|^2 ds, \int_0^t |Z_s|^2 ds \right) \geq n \right\} \wedge T,$$

and the probability

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} = M_{\tau_n}, \quad \text{with} \quad M_t = \exp \left( \int_0^t q_s dW_s - \frac{1}{2} \int_0^t |q_s|^2 ds \right).$$

We will show that  $(M_{\tau_n})_{n \in \mathbb{N}}$  is uniformly integrable which is sufficient to conclude. Since  $(Y, Z)$  solves the BSDE (1.1), we have

$$\begin{aligned} Y_0 &= Y_{\tau_n} - \int_0^{\tau_n} g(Z_s) ds + \int_0^{\tau_n} Z_s dW_s \\ &= Y_{\tau_n} + \int_0^{\tau_n} (Z_s q_s - g(Z_s)) ds + \int_0^{\tau_n} Z_s (dW_s - q_s ds) \\ &= \mathbb{E}^{\mathbb{Q}_n} \left[ Y_{\tau_n} + \int_0^{\tau_n} g^*(q_s) ds \right]. \end{aligned} \quad (3.3)$$

Firstly, since  $\Psi$  and  $\Phi$  are dual functions, the Fenchel's inequality gives us

$$\mathbb{E}^{\mathbb{Q}_n} [Y_{\tau_n}] \geq -\mathbb{E}^{\mathbb{Q}_n} [Y_{\tau_n}^-] \geq -\mathbb{E} [\Psi(Y_{\tau_n}^-)] - \mathbb{E} [\Phi(M_{\tau_n})].$$

Moreover, we have, thanks to (3.1),

$$-\mathbb{E} [\Psi(Y_{\tau_n}^-)] \geq -\mathbb{E} [K(Y_{\tau_n}^-)] \geq -C,$$

with  $C$  a constant that does not depend on  $n$ . By putting these inequalities into (3.3) we obtain

$$Y_0 \geq -C - \mathbb{E}[\Phi(M_{\tau_n})] + \mathbb{E}^{\mathbb{Q}_n} \left[ \int_0^{\tau_n} g^*(q_s) ds \right]. \quad (3.4)$$

Thanks to the growth of  $g^*$  given by (3.2) we have

$$\mathbb{E}^{\mathbb{Q}_n} \left[ \int_0^{\tau_n} g^*(q_s) ds \right] \geq -C + \mathbb{E}^{\mathbb{Q}_n} \left[ \frac{1}{2\gamma} \int_0^{\tau_n} |q_s|^2 ds \right].$$

Moreover, a simple calculus gives us

$$\mathbb{E}^{\mathbb{Q}_n} \left[ \frac{1}{2\gamma} \int_0^{\tau_n} |q_s|^2 ds \right] = \frac{1}{\gamma} \mathbb{E} [M_{\tau_n} \ln(M_{\tau_n})].$$

By putting these two results into (3.4), and by setting  $\Lambda = \Phi_0 - \Phi$ , we obtain

$$Y_0 \geq -C + \mathbb{E}[\Lambda(M_{\tau_n})] - \mathbb{E}[\Phi_0(M_{\tau_n})] + \frac{1}{\gamma} \mathbb{E} [M_{\tau_n} \ln(M_{\tau_n})]. \quad (3.5)$$

Let us remark that

$$\begin{aligned} & \mathbb{E}[\Phi_0(M_{\tau_n})] - \frac{1}{\gamma} \mathbb{E} [M_{\tau_n} \ln(M_{\tau_n})] \\ &= \mathbb{E} \left[ \frac{M_{\tau_n}}{\gamma} \ln \left( 1 + \frac{\gamma}{M_{\tau_n}} \right) \right] - \left( \frac{\ln \gamma + 1}{\gamma} \right) \mathbb{E} [M_{\tau_n}] + \mathbb{E} \left[ \ln \left( \frac{M_{\tau_n}}{\gamma} + 1 \right) \right] \\ &= \mathbb{E} \left[ \frac{M_{\tau_n}}{\gamma} \ln \left( 1 + \frac{\gamma}{M_{\tau_n}} \right) \right] - \left( \frac{\ln \gamma + 1}{\gamma} \right) + \mathbb{E} \left[ \ln \left( \frac{M_{\tau_n}}{\gamma} + 1 \right) \right]. \end{aligned}$$

An elementary inequality gives us

$$\mathbb{E} \left[ \frac{M_{\tau_n}}{\gamma} \ln \left( 1 + \frac{\gamma}{M_{\tau_n}} \right) \right] \leq \mathbb{E} \left[ \frac{M_{\tau_n}}{\gamma} \frac{\gamma}{M_{\tau_n}} \right] \leq 1,$$

and

$$\mathbb{E} \left[ \ln \left( \frac{M_{\tau_n}}{\gamma} + 1 \right) \right] \leq \mathbb{E} \left[ \frac{M_{\tau_n}}{\gamma} \right] \leq \frac{1}{\gamma}.$$

Thus, we have

$$\mathbb{E}[\Phi_0(M_{\tau_n})] - \frac{1}{\gamma} \mathbb{E} [M_{\tau_n} \ln(M_{\tau_n})] \leq C$$

and inequality (3.5) becomes

$$Y_0 \geq -C + \mathbb{E}[\Lambda(M_{\tau_n})]. \quad (3.6)$$

Let us give a useful property of  $\Lambda$  that we will prove after.

**Proposition 3.3** *The function  $\Lambda$  satisfies*

$$\lim_{x \rightarrow +\infty} \frac{\Lambda(x)}{x} = +\infty.$$

Thanks to this proposition and the inequality (3.6) we are allowed to apply the de la Vallée-Poussin Theorem:  $(M_{\tau_n})_{n \in \mathbb{N}}$  is uniformly integrable and the proof is finished.  $\square$

**Proof of Proposition 3.3:** It is sufficient to show that  $\Lambda' = \Phi'_0 - \Phi'$  is increasing and  $\lim_{x \rightarrow +\infty} \Lambda'(x) \rightarrow +\infty$ . Firstly, let us show that  $\Psi''(\Phi'(x)) \geq \gamma(x + \gamma)$ , for all  $x \geq 0$ :

$$\Psi''(x) = k'(x)(e^{\gamma x} - 1) + k(x)\gamma e^{\gamma x} \geq \gamma k(x)(e^{\gamma x} - 1) + \gamma k(x) \geq \gamma \Psi'(x) + \gamma^2,$$

so we have

$$\Psi''(\Phi'(x)) \geq \gamma \Psi'(\Phi'(x)) + \gamma^2 = \gamma(x + \gamma).$$

As a result, we get from the equality  $(\Psi'(\Phi'(x)))' = \Psi''(\Phi'(x))\Phi''(x) = 1$  that  $\Phi''(x) \leq \frac{1}{\gamma(x + \gamma)}$ . We finally obtain

$$\Lambda''(x) = \Phi''_0(x) - \Phi''(x) \geq \frac{1}{\gamma(x + \gamma)} - \frac{1}{\gamma(x + \gamma)} \geq 0$$

and so  $\Lambda'$  is an increasing function. To conclude we will prove by contradiction that  $\Lambda'$  is an unbounded function: let us assume that there exists a constant  $A$  such that  $\Lambda' \leq A$ . Then we have

$$\begin{aligned} x &= \Psi'(\Phi'(x)) = k(\Phi'(x)) \left( e^{\gamma\Phi'(x)} - 1 \right) = k(\Phi'(x)) \left( e^{\gamma(\Phi'_0(x) - \Lambda'(x))} - 1 \right) \\ &\geq k(\Phi'(x)) \left( e^{\gamma\Phi'_0(x)} e^{-A} - 1 \right) \geq k(\Phi'(x)) \left( \left( \frac{x}{\gamma} + 1 \right) e^{-A} - 1 \right), \end{aligned}$$

and so, we get for  $x$  big enough

$$k(\Phi'(x)) \leq \frac{x}{\left( \frac{x}{\gamma} + 1 \right) e^{-A} - 1} \leq C.$$

Since  $\lim_{x \rightarrow +\infty} \Phi'(x) = +\infty$ , previous inequality gives us that  $k$  is a bounded function, which is a contradiction.  $\square$

**Remark 3.4** In [3], the authors proved that if for some  $p > \gamma$ ,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{pY_t^-} \right] < \infty,$$

then  $\mathcal{E}(q)$  has finite entropy, i.e.,

$$\mathbb{E} [\mathcal{E}(q)_T \ln \mathcal{E}(q)_T] < +\infty.$$

However, in the critical case,  $e^{-\gamma Y}$  belongs to the class (D), this property is not always true. It suffices to take again  $g(z) = \frac{\gamma}{2} |z|^2$ , then if  $Y \leq 0$ ,

$$\mathcal{E}(q)_t \ln \mathcal{E}(q)_t = e^{\gamma Y_0} e^{\gamma Y_t^-} (\gamma Y_0 + \gamma Y_t^-).$$

It follows that if  $g(z) = \frac{\gamma}{2} |z|^2$  and  $Y \leq 0$ ,  $\mathcal{E}(q)$  has finite entropy if and only if  $Y^- e^{\gamma Y^-}$  belongs to the class (D).

## 4 The uniqueness result

Remark 3.4 indicates that  $\mathcal{E}(q)$  does not always have finite entropy in the critical case. Hence we could not adopt the verification argument given in [3] to show the uniqueness. In this last section, we show the uniqueness under the additional assumption B.

**Theorem 4.1** Let us assume that assumptions A and B hold true. Then the BSDE (1.1) has a unique solution  $(Y, Z)$  such that  $Y$  and  $e^{-\gamma Y}$  belong to the class (D).

**Proof of Theorem 4.1.** The existence result is already given in Theorem 2.2. For the uniqueness, let us consider  $(Y, Z)$  and  $(Y', Z')$  two solutions of the BSDE (1.1) such that  $Y, Y', e^{-\gamma Y}$  and  $e^{-\gamma Y'}$  belong to the class (D). By a symmetry argument it is sufficient to show that  $Y_t \geq Y'_t$   $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ . For  $t \in [0, T]$ , let us denote  $A := \{Y_t < Y'_t\}$  and set the stopping time  $\tau = \inf \{s \geq t | Y_s \geq Y'_s\}$ . Then, for  $s \in [t, \tau]$  we have  $Y_s \leq Y'_s$  and  $Y_\tau = Y'_\tau$   $\mathbb{P}$ -a.s. because  $t \rightarrow Y_t$  is continuous  $\mathbb{P}$ -a.s..

Let us consider a predictable process  $(q_s)_{s \in [0, T]}$  with values in  $\mathbb{R}^d$  and such that  $q_s \in \partial g(Z_s)$  for all  $s \in [0, T]$ . Thanks to Proposition 3.1 we know that  $\mathcal{E}(q)$  defines a probability that we will denote  $\mathbb{Q}$ . Under  $\mathbb{Q}$ , we get

$$d(Y_s - Y'_s) = (g(Z_s) - g(Z'_s) - (Z_s - Z'_s)q_s)ds - (Z_s - Z'_s)dW_s^{\mathbb{Q}}. \quad (4.1)$$

Then, Itô formula gives us, for  $0 < \alpha \leq \varepsilon$ ,

$$\begin{aligned} &de^{\alpha(Y_{s \wedge \tau} - Y'_{s \wedge \tau} + C_2(T-s))} \mathbb{1}_A \\ &= -\alpha \mathbb{1}_A e^{\alpha(Y_{s \wedge \tau} - Y'_{s \wedge \tau} + C_2(T-s))} \mathbb{1}_A \left( g(Z'_s) - g(Z_s) - (Z'_s - Z_s)q_s + C_2 - \frac{\alpha}{2} |Z'_s - Z_s|^2 \right) \mathbb{1}_{s \leq \tau} ds \\ &\quad - \alpha \mathbb{1}_A e^{\alpha C_2(T-s)} \mathbb{1}_A \mathbb{1}_{s > \tau} ds + \mathbb{1}_{s \leq \tau} dM_s, \end{aligned}$$

with  $(M_s)_{s \in [t, \tau]}$  a local martingale under  $\mathbb{Q}$ . From assumption B we have that

$$g(Z'_s) - g(Z_s) - (Z'_s - Z_s)q_s \geq \frac{\varepsilon}{2} |Z'_s - Z_s|^2 - C_2.$$

So, we obtain that  $\left(e^{\alpha(Y_{s\wedge\tau}-Y'_{s\wedge\tau}+C_2(T-s))}\mathbb{1}_A\right)_{t\leq s\leq T}$  is a bounded supermartingale under  $\mathbb{Q}$  and

$$e^{\alpha(Y_{s\wedge\tau}-Y'_{s\wedge\tau}+C_2(T-s))}\mathbb{1}_A \geq \mathbb{E}\left[e^{\alpha(Y_\tau-Y'_\tau+C_2(T-T))}\mathbb{1}_A\middle|\mathcal{F}_s\right] = 1, \quad \forall s \in [t, T].$$

It implies that  $((Y_{s\wedge\tau} - Y'_{s\wedge\tau})\mathbb{1}_A)_{s\in[t,T]}$  is a bounded process. Moreover,  $g$  is a convex function so

$$g(z) - g(z') - (z - z')u \leq 0, \quad \forall z, z' \in \mathbb{R}^{1\times d}, \quad \forall u \in \partial g(z).$$

By using this inequality in (4.1), we obtain that  $((Y_{s\wedge\tau} - Y'_{s\wedge\tau})\mathbb{1}_A)_{s\in[t,T]}$  is a bounded negative supermartingale under  $\mathbb{Q}$  such that  $(Y_\tau - Y'_\tau)\mathbb{1}_A = 0$ . We conclude that  $(Y_t - Y'_t)\mathbb{1}_A = 0$ , that is to say,  $Y_t \geq Y'_t$ . Finally, it is rather standard to show that  $\int_0^T |Z_s - Z'_s|^2 ds = 0$   $\mathbb{P}$ -a.s..  $\square$

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